Unifying Characterization of Max-Min Fairness in Wireless Networks by Graphs

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***Abstract*— We propose a unifying framework for max-min fairness in orthogonal networks and networks with interference. First, a universal formulation of the max-min fairness problem for orthogonal networks and networks with interference is presented. This shows that orthogonal networks and networks with interference can be universally described by a graph, induced by time sharing of resources and interference coupling, respectively. As a consequence, a unifying graph-related characterization of performance achieved under max-min fairness is obtained.**

I. Introduction

Fairness represents an important goal in the design of resource allocation policies for wireless networks. In multi-hop ad-hoc networks, it is usually desired to achieve fairness with respect to performance metrics (such as data-rate, or bit-error-rate) associated with end-to-end routes [1], [2]. In single hop communication, fairness is desired with respect to link performance metrics. The mostly used notion of fairness is the max-min fairness, which corresponds to ideal social fairness and consists in the maximal possible improvement of the worst performance metric [3].

For networks with interference, the issue of max-min fairness has been addressed frequently. Networks with interference include cellular networks with multi-antenna base-station using receive/ transmit beamforming, and cellular CDMA (*Code Division Multiple Access*) networks as the most prominent examples [4], [5]. In such networks the interest is usually in max-min fairness with respect to link SIR values (*Signal to Interference Ratio*) relative to the given link SIR requirements. A related form of such max-min fairness problem is the minimization of total transmit power, provided that the SIR values exceed the corresponding SIR requirements [4], [6], [7]. In this widely-studied problem, the optimal power allocation solves a linear matrix equation and the link performance can be characterized by means of Perron-Frobenius Theory [3], [6], [7], [8]. However, not much is known on the link performance achieved under max-min fairness when the power allocation is optimized jointly with the transmit/ receive policy, that is, with transmit/ receive beamforming vectors, or correlation receivers and spreading sequences (CDMA). See e.g. [5], [6], [7] for insights into such more general max-min fairness problem.

For orthogonal networks, the issue of max-min fairness has been addressed in [9], [10] [11] (and references therein). Concurrently, a cellular OFDM (*Orthogonal Frequency Division Multiplex*) network appears to be the most relevant example of an orthogonal network. In [9], the max-min fair carrier and antenna assignment is studied for a multi-antenna OFDM network. More generally, in [10] the performance of joint max-min fair power allocation and orthogonal resource assignment is characterized within the framework of so-called blocking and antiblocking polyhedra. The characterization of link performance achieved under max-min fairness provided in [10], is however in form of bounds and a duality-like optimization problem.

In this work we aim at providing a unifying characterization of link performance achieved under max-min fairness for wireless networks with interference, such as CDMA, and wireless orthogonal networks, such as OFDM. According to the above, such characterization is, to the best of our knowledge, not known in satisfactory generality.

i.) We obtain a unifying formulation of the max-min fairness problem for (single-hop) networks with interference and orthogonal networks (Section III).

ii.) We propose a universal description of networks by graphs, which is suggested by the obtained unifying problem formulation (Section IV). In the case of orthogonal networks, such graph is induced by the time sharing of orthogonal resources. In networks with interference, the graph is determined by the interference coupling.

iii.) As a consequence, we obtain the main result in form of unifying inequality and equality characterizations of link performance achieved under max-min fairness in networks with interference and in orthogonal networks (Sections V, VI).

II. Models and Preliminaries

We consider a general single-hop wireless network model. The set of links is denoted by $\mathcal{K} = \{1, \ldots, K\}$. The set of medium resources in the network is denoted by $\mathcal{N} = \{1, \ldots, N\}$. We do not regard transmit power as a medium resource. We address two main types of networks.

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i.) Networks with interference, or simply interference networks: For interference networks we assume that the medium resources $n \in \mathcal{N}$ are nonorthogonal, i.e. the links accessing the wireless medium interfere. For instance, in a CDMA network the wireless medium is accessed concurrently by links $k \in \mathcal{K}$, which utilize different spreading sequences [6]. In a cellular downlink/ uplink with multi-antenna base station, the links $k \in \mathcal{K}$ utilize different spatial directions of transmission/ reception, described by Minimum Mean Square Error (MMSE) transmit/ receive beamformers [5], [7]. The spreading sequences and beamformers correspond to nonorthogonal medium resources $n \in \mathcal{N}$.

ii.) Orthogonal networks: For orthogonal networks we assume that the medium resources are orthogonal, i.e. the links assigned some medium resources do not interfere with each other. For instance, in an OFDM network the wireless medium is accessed concurrently by links $k \in \mathcal{K}$, which are assigned ensembles of OFDM carriers. The OFDM carriers correspond to orthogonal medium resources $n \in \mathcal{N}$.

We assume throughout the work $N \geq K$. This is justified for CDMA and OFDM networks, in which the number of spreading sequences and carriers, respectively, is usually no smaller than the number of links. We denote by matrix $\boldsymbol{P} = (p_{kn}) = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K)' \in \mathbb{R}_+^{K \times N}$ the power allocation, that is, the matrix of transmit powers allocated to pairs of links and resources. Thus, vector $\boldsymbol{p}_k$ contains the powers used by link k on resources $n \in \mathcal{N}$. We assume one of the sets

$$\mathcal{P}_s = \{\boldsymbol{P} \in \mathbb{R}_+^{K \times N} : |||\boldsymbol{P}|||_1 \leq P\},$$
$$\mathcal{P}_i = \{\boldsymbol{P} \in \mathbb{R}_+^{K \times N} : \|\boldsymbol{p}_k\|_1 \leq P_k, k \in \mathcal{K}\}$$

as the set of allowable power allocations.

The set $\mathcal{P}_s$ can correspond to the cellular downlink, where sum-power of the base station is constrained. The set $\mathcal{P}_i$ corresponds to single-hop ad-hoc networks or cellular uplink, where each link power is constrained. Sometimes a universal notation $\mathcal{P} \in \{\mathcal{P}_s, \mathcal{P}_i\}$ is used. The requirements of links $k \in \mathcal{K}$ with respect to considered link performance metrics are grouped in the vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K) \in \mathbb{R}_{++}^K$.

A. Orthogonal Networks

In orthogonal networks, the set of resources $\mathcal{N}$ is shared over time by the links $k \in \mathcal{K}$. Each link-resource pair $(k, n) \in \mathcal{K} \times \mathcal{N}$ is parameterized by its fixed channel and noise variance value. For orthogonal networks we introduce general performance functions

$$\boldsymbol{P} \mapsto f_k(\boldsymbol{P}) = (f_{k1}(\boldsymbol{P}), \ldots, f_{kN}(\boldsymbol{P})), \quad \boldsymbol{P} \in \mathcal{P}, \quad k \in \mathcal{K}.$$

The vector-valued function f_k contains the performance metrics of link $k \in \mathcal{K}$, as functions of power allocation, associated with resources $n \in \mathcal{N}$. We require f_k, $k \in \mathcal{K}$ to satisfy

Conditions 1:
i.) $f_k(\boldsymbol{P}) \in \mathbb{R}_+^N$, $\boldsymbol{P} \in \mathcal{P}$, $k \in \mathcal{K}$ *(f_k is nonnegative),*
ii.) f_{kn} *is decreasing in* p_{kn}, $(k, n) \in \mathcal{K} \times \mathcal{N}$.

Examples of such performance functions are;

i.) Bit-Error-Rate (BER) function, for which we have (under most relevant modulation schemes and Gaussian noise)

$$f_{kn}(\boldsymbol{P}) = Q((c_{kn}|h_{kn}|^2 p_{kn})^{\frac{1}{2}}), \tag{1}$$

with Q as the Marcum Q-function, h_{kn} as channel of link k on resource n and noise-dependent constant c_{kn}, $(k, n) \in \mathcal{K} \times \mathcal{N}$.

ii.) MMSE function, for which we have (under the MMSE receiver)

$$f_{kn}(\boldsymbol{P}) = (1 + b_{kn}|h_{kn}|^2 p_{kn})^{-1}, \tag{2}$$

with noise-dependent constant b_{kn}, $(k, n) \in \mathcal{K} \times \mathcal{N}$. Due to orthogonality we have for the most relevant metrics $f_{kn}(\boldsymbol{P}) = f_{kn}(p_{kn})$, $(k, n) \in \mathcal{K} \times \mathcal{N}$, as is the cases of (1), (2).

We denote by the matrix $\boldsymbol{A} = (a_{kn}) = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_K)' \in \mathbb{R}_+^{K \times N}$ the time sharing matrix, that is, the matrix of relative time fractions assigned to links $k \in \mathcal{K}$ for the separate use of single resources $n \in \mathcal{N}$. Thus, vector $\boldsymbol{a}_k$ groups the relative time fractions which are assigned to link k for the exclusive use of resources $n \in \mathcal{N}$. We assume the set of allowable time sharing matrices as

$$\mathcal{A}(\boldsymbol{r}) = \{\boldsymbol{A} \in \mathbb{R}_+^{K \times N} : \|\boldsymbol{a}_k\|_1 = r_k, k \in \mathcal{K}, \|\boldsymbol{r}\|_1 = N, \sum_{k \in \mathcal{K}} a_{kn} \leq 1, n \in \mathcal{N}\}, \quad \boldsymbol{r} = (r_1, \ldots, r_K) \in \mathbb{R}_{++}^K.$$

Precisely, $\boldsymbol{A} \in \mathcal{A}(\boldsymbol{r})$ means that link k uses the fraction r_k/N of all resources on average over time. The parameter vector $\boldsymbol{r}/N$ contains the fractions of $\mathcal{N}$ assigned to links $k \in \mathcal{K}$ over time. For instance, under $\boldsymbol{r} = \boldsymbol{1}$ each link is assigned an equal fraction N/K of resources over time.

B. Interference Networks

In interference networks each link is usually assigned a single nonorthogonal resource and we restrict us here to such case. Thus, only K from N nonorthogonal resources are assigned to links $k \in \mathcal{K}$ As a consequence, only one entry in each row of $\boldsymbol{P}$ is nonzero, and we assume without loss of generality that this is the entry p_{kk}, $k \in \mathcal{K}$.

The link SIR is the link performance metric of interest in interference networks, since most relevant measures of perceived end-performance can be expressed as monotone functions of the link SIR [3], [7], [8]. This includes the data rate, and approximately also the bit-error-rate under soft-decision decoding [12].

Under linear receivers, the link SIR can be written as [4], [7]

$$\mathrm{SIR}_k(\boldsymbol{B}, \boldsymbol{P}) = \frac{p_{kk}}{\sum_{l \in \mathcal{K}\, ,\, l \neq k} b_{kl}|h_{kl}|^2/|h_{kk}|^2 p_{ll} + \sigma_k^2}, \tag{3}$$

where h_{kl} denotes the coefficient of the channel from the transmitter of link $l \in \mathcal{K}$ to the receiver of link k, and σ_k^2 denotes the Gaussian noise variance at the output of receiver of link $k \in \mathcal{K}$. Matrix $\boldsymbol{B} = [b_{kl}] = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_K)' \in \mathbb{R}_+^{K \times K}$, is referred to as interference matrix and contains pairwise coefficients of interference attenuation in the network. Thus, vector $\boldsymbol{b}_k$ contains the coefficients of interference from links $l \in \mathcal{K}$, $l \neq k$ perceived at the receiver of link k and the element b_{kk}, which can be set arbitrarily. For our purposes it is convenient to set $b_{kk}=1$.

For a synchronous CDMA network, such as CDMA downlink, it is known that the interference matrix can be expressed precisely as $\boldsymbol{B}=[|\langle \boldsymbol{c}_k, \boldsymbol{s}_l\rangle|^2]$, with $\boldsymbol{s}_k, \boldsymbol{c}_k \in \mathbb{R}^M$, $k\in\mathcal{K}$ as spreading sequences and correlation receivers, respectively [6].

To account for constraints on the design of beamformers or correlation receivers and spreading sequences, we assume the inclusion of the interference matrix in

$$\mathcal{B}(\boldsymbol{r})=\{\boldsymbol{B}\in\mathbb{R}_+^{K\times K}: b_{kk}=1, \|\boldsymbol{b}_k\|_1 \le r_k, k\in\mathcal{K}, \\ \|\boldsymbol{b}_j\|_1 = r_j \text{ for some } j\in\mathcal{K}\},\ \boldsymbol{r}=(r_1,\ldots,r_K)\in\mathbb{R}_{++}^K.$$

Thereby, the parameter vector $\boldsymbol{r}\in\mathbb{R}_{++}^K$ can be set arbitrarily in order to to account for a large class of beamformer designs and correlation receiver and spreading sequence designs (see e.g. [6] for the case of certain families of spreading sequences).

III. Unifying Form of the Fairness Problem

We consider max-min fairness as the desired notion of fairness to be ensured.

Definition 1: Given link performance requirements $\boldsymbol{\gamma}$, we say that (relative) max-min fairness is achieved in the network if the worst-case ratio of link performance value and corresponding requirement γ_k is maximally improved.

For orthogonal networks, we assumed general link performance functions f_k, $k\in\mathcal{K}$, satisfying Conditions 1. Thus, given $\boldsymbol{r}$ and link performance requirements $\boldsymbol{\gamma}$, the worst-case link performance aggregated over all orthogonal resources, relative to the corresponding requirement is $\max_{k\in\mathcal{K}}\langle \boldsymbol{a}_k, f_k(\boldsymbol{P})\rangle/\gamma_k$, with $(\boldsymbol{A},\boldsymbol{P})\in\mathcal{A}(\boldsymbol{r})\times\mathcal{P}$, which has to be minimized in order to achieve max-min fairness. Thus, inverting the ratio, the max-min fairness problem for orthogonal networks can be formulated as

$$\max_{(\boldsymbol{V},\boldsymbol{P})\in\mathcal{V}(\boldsymbol{r})\times\mathcal{P}} \min_{k\in\mathcal{K}} \frac{\gamma_k}{\langle \boldsymbol{v}_k, f_k(\boldsymbol{P})\rangle}, \quad \boldsymbol{r}\in\mathbb{R}_{++}^K, \tag{4}$$

with functions f_k, $k\in\mathcal{K}$, satisfying Conditions 1 and

$$\mathcal{V}(\boldsymbol{r})=\mathcal{A}(\boldsymbol{r}), \quad \boldsymbol{\gamma}\in\mathbb{R}_{++}^K. \tag{5}$$

In interference networks, the link SIR is the desired link performance metric. Thus, given some $\boldsymbol{r}$ and link SIR requirements $\hat{\boldsymbol{\gamma}}$, the value $\min_{k\in\mathcal{K}}\mathrm{SIR}_k(\boldsymbol{B},\boldsymbol{P})/\hat{\gamma}_k$, with $(\boldsymbol{B},\boldsymbol{P})\in\mathcal{B}(\boldsymbol{r})\times\mathcal{P}$, represents the worst-case link performance relative to the corresponding requirement. Hence, we can write the max-min fairness problem in interference networks as

$$\max_{(\boldsymbol{B},\boldsymbol{P})\in\mathcal{B}(\boldsymbol{r})\times\mathcal{P}} \min_{k\in\mathcal{K}} \frac{\mathrm{SIR}_k(\boldsymbol{B},\boldsymbol{P})}{\hat{\gamma}_k}, \quad \boldsymbol{r},\hat{\boldsymbol{\gamma}}\in\mathbb{R}_{++}^K. \tag{6}$$

A key observation is that problem (6) is equivalent to a particular instance of problem (4).

Proposition 1: The max-min fairness problem (6) for an interference network is equivalent to problem (4), with functions f_k, $k\in\mathcal{K}$, such that for some $\epsilon=\epsilon(\hat{\boldsymbol{\gamma}})>0$

$$f_{kk}(\boldsymbol{P})=\begin{cases}\frac{\sigma_k^2}{p_{kk}} & p_{kk}\ge\epsilon\\ \frac{\sigma_k^2}{\epsilon} & p_{kk}<\epsilon,\end{cases} \quad f_{kl}(\boldsymbol{P})=\begin{cases}\frac{|h_{kl}|^2 p_{ll}}{|h_{kk}|^2 p_{kk}} & p_{kk}\ge\epsilon\\ \frac{|h_{kl}|^2 p_{ll}}{|h_{kk}|^2 \epsilon} & p_{kk}<\epsilon,\end{cases} \tag{7}$$

$k\neq l$, and with

$$\mathcal{V}(\boldsymbol{r})=\mathcal{B}(\boldsymbol{r}), \quad \boldsymbol{\gamma}=(\frac{1}{\hat{\gamma}_1},\ldots,\frac{1}{\hat{\gamma}_K}). \tag{8}$$

It can be seen that the functions f_k, $k\in\mathcal{K}$, defined in (7) satisfy Conditions 1 for $K=N$. Thus, by Proposition 1, the max-min fairness problem (6) for an interference network is a particular instance of the max-min fairness problem for an orthogonal network. Equivalently, (4) is a unifying form of the max-min fairness problem for orthogonal and interference networks. The particular problem version for interference networks is yielded from (4) by

i.) setting $K=N$ and choosing the particular performance functions f_k, $k\in\mathcal{K}$, defined in (7),

ii.) setting (8), which includes taking the link performance requirements as inverse SIR requirements.

IV. Graph View of Time Sharing and Interference

The general network type-insensitive form of the fairness problem (4) suggests us the use of a network type- insensitive matrix $\boldsymbol{V}\in\mathcal{V}(\boldsymbol{r})$, $N\ge K$, as a means of network description. We can refer to such universal matrix simply as a network matrix. A network matrix represents some time sharing matrix from $\mathcal{V}(\boldsymbol{r})=\mathcal{A}(\boldsymbol{r})$, $\boldsymbol{r}\in\mathbb{R}_{++}^K$, if an orthogonal network is considered, or it corresponds to an interference matrix from $\mathcal{V}(\boldsymbol{r})=\mathcal{B}(\boldsymbol{r})$, $\boldsymbol{r}\in\mathbb{R}_{++}^K$, in the case of an interference network.

For a network matrix we can define a simple relation between its rows, by distinguishing the cases

$$\begin{matrix}\langle \boldsymbol{v}_k, \boldsymbol{v}_l\rangle > 0,\\ \langle \boldsymbol{v}_k, \boldsymbol{v}_l\rangle = 0,\end{matrix} \quad k\neq l, \quad (k,l)\in\mathcal{K}^2. \tag{9}$$

Relations (9) give rise to the definition of the network graph which is induced by a network matrix $\boldsymbol{V}\in\mathcal{V}(\boldsymbol{r})$. Such (undirected) graph is defined in the usual way, as a set of graph nodes $\mathcal{K}$ and a set of graph edges, with an edge represented as a pair of nodes $(k,l)\in\mathcal{K}^2$ which it connects. We say that a node pair $(k,l)\in\mathcal{K}^2$ in the graph is adjacent iff (k,l) is an edge (pertains to the set of edges) of the graph [13].

The manner in which the row relations (9) of a network matrix $\boldsymbol{V}\in\mathcal{V}(\boldsymbol{r})$ induce the corresponding network graph was originally proposed in Lovasz's seminal work [14].

Definition 2: The network graph $G(\boldsymbol{V})$ defined on the node set $\mathcal{K}$ and determined by a network matrix $\boldsymbol{V}\in\mathcal{V}(\boldsymbol{r})$ is such that a node pair (k,l) is adjacent in $G(\boldsymbol{V})$ iff $\langle \boldsymbol{v}_k, \boldsymbol{v}_l\rangle > 0$. Network matrix $\boldsymbol{V}$ is referred then as an orthonormal representation of the network graph $G(\boldsymbol{V})$.

The obtained network graph $G(\boldsymbol{V})$ is not only a universal characterization of orthogonal and interference networks. It is additionally a combinatorial characterization, that is, it is determined solely by binary relations of adjacency and nonadjacency of its nodes.

A network graph is easily interpretable in terms of time sharing and interference. For orthogonal networks, adjacency of the node pair (k,l) in the network graph or, equivalently, inequality in (9), means that there is a resource $n\in\mathcal{N}$ which is shared by both links k,l. Hereby, we say that a resource is shared by links k, l if it is used separately by links k and l some fraction of time. For interference networks, adjacency of the node pair (k, l) in the graph $G(\boldsymbol{V})$, or, equivalently, inequality in (9), means that either there is a link $j \in \mathcal{K}$, $j \neq l$, $j \neq k$, which causes interference at the receivers of both links k, l, or one of the links in the pair (k, l) causes interference at the receiver of the other link in the pair[1].

Analogously, nonadjacency of the pair (k, l) in the network graph means in orthogonal networks that link k does not share any common resource with link l. Nonadjacency of the pair (k, l) in the graph $G(\boldsymbol{V})$ in interference networks means that the receivers of links k, l do not have a common interferer $j \in \mathcal{K}$ and none of the links in the pair (k, l) causes interference at the receiver of the other link in the pair. Examples of an orthogonal and interference network which have the same network graph are depicted in Fig. 1.

A. *The Lovasz Θ-Function*

A network graph $G(\boldsymbol{V})$ can be associated the value of the weighted Lovasz Θ-function $\Theta(G(\boldsymbol{V}), \boldsymbol{x})$, with some weight vector $\boldsymbol{x} \in \mathbb{R}_+^K$. The weighted Lovasz Θ-function (in short, Θ-function) is defined on graphs and vectors and was introduced by Lovasz (in unweighted form) in [14]. The Θ-function was used originally to characterize the graph capacity in the framework of zero-error information theory. Numerous analytical representations of the Θ-function are known [15]. For the proofs of our results, we need the following one.

Proposition 2: Let $G = G(\boldsymbol{V})$ be the network graph induced by a network matrix $\boldsymbol{V} \in \mathcal{V}(\boldsymbol{r})$. Then, we have

$$\Theta(G, \boldsymbol{w}^2) = \min_{\boldsymbol{C} \in \mathcal{C}(G, \boldsymbol{w})} \lambda_{max}(\boldsymbol{C}), \quad \boldsymbol{w} = (w_1, \ldots, w_K) \in \mathbb{R}_+^K,$$

with $\boldsymbol{w}^2 = (w_1^2, \ldots, w_K^2)$ and the class of matrices

$$\mathcal{C}(G, \boldsymbol{w}) = \{\boldsymbol{C} = (c_{kl}) \in \mathbb{R}^{K \times K} : c_{kl} = w_k w_l \text{ if } (k, l),\ k \neq l \text{ is nonadjacent in } G, \text{ or } k = l\}.$$

In other words, the value of the Θ-function of a network graph $G(\boldsymbol{V})$ and weight vector $\boldsymbol{w}^2$ is simply the minimum spectral radius achieved in the class of (symmetric) matrices which indicate the nonadjacency of any two nodes (k, l) in the network graph by the value $w_k w_l$.

V. Unifying Description of Fairness Performance

The obtained common problem form (4) and the results of Section IV indicate that the problem of max-min fairness is related to some network type-insensitive graph notion. In broad terms, this is precisely our first main result.

Proposition 3: Let $G = G(\boldsymbol{V})$ be the network graph induced by a network matrix $\boldsymbol{V} \in \mathcal{V}(\boldsymbol{r})$, let f_k, $k \in \mathcal{K}$, be bounded and satisfy Conditions 1 and define

$$(\hat{\boldsymbol{V}}, \hat{\boldsymbol{P}}) = \arg \max_{(\boldsymbol{V}, \boldsymbol{P}) \in \mathcal{V}(\boldsymbol{r}) \times \mathcal{P}} \min_{k \in \mathcal{K}} \frac{\gamma_k}{\langle \boldsymbol{v}_k, f_k(\boldsymbol{P}) \rangle},$$

[1]Notice from (9) and the definition of $\mathcal{B}(\boldsymbol{r})$ that the latter case is a consequence of the assumed positivity of the diagonal elements of any interference matrix. By setting the diagonal elements to zero, the topology of the network graph changes, but all the results of this work remain true.

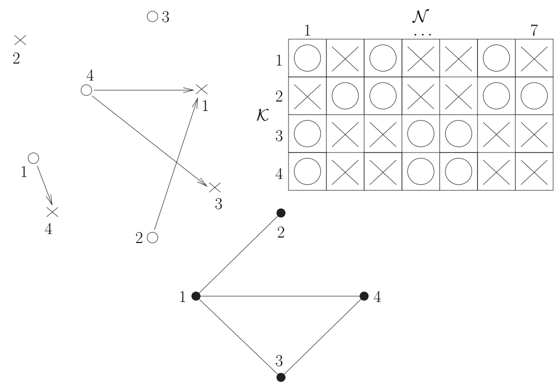

Fig. 1. On the left-hand side we have an interference network with $K = 4$, with link transmitters and receivers labeled by $\circ$ and $\times$, respectively, and the corresponding link number. An arrow from link transmitter l to link receiver k denotes interference (i.e. $b_{kl} > 0$). On the right-hand side we have a mask of a time sharing matrix of an orthogonal network with $K = 4, N = 7$. The (k, n)-th entry in the mask is labeled by $\circ$ if link k uses resource n some fraction of time. Otherwise the entry is labeled by $\times$. The resulting network graph, common for both networks, is depicted at the bottom.

with $\mathcal{V}(\boldsymbol{r}) \in \{\mathcal{A}(\boldsymbol{r}), \mathcal{B}(\boldsymbol{r})\}$, $\boldsymbol{r} \in \mathbb{R}_{++}^K$. Then,

$$\frac{\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)}{c} - \delta \leq \max_{\substack{(\boldsymbol{V}, \boldsymbol{P}) \in \\ \mathcal{V}(\boldsymbol{r}) \times \mathcal{P}}} \min_{k \in \mathcal{K}} \frac{\gamma_k^2}{\langle \boldsymbol{v}_k, f_k(\boldsymbol{P}) \rangle^2} \leq \frac{\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)}{c}, \tag{10}$$

with

$$\begin{cases} \boldsymbol{w} = \boldsymbol{w}(\boldsymbol{r}, \boldsymbol{\gamma}) = (\gamma_1 / r_1, \ldots, \gamma_K / r_K) \\ c = c(\hat{\boldsymbol{P}}) = \max_{k \in \mathcal{K}} \langle f_k(\hat{\boldsymbol{P}}), f_k(\hat{\boldsymbol{P}}) \rangle, \end{cases} \tag{11}$$

and some $\delta = \delta(\hat{\boldsymbol{P}}, \hat{\boldsymbol{V}}, \mathcal{P}, \mathcal{V}(\boldsymbol{r})) \geq 0$ expressed as

$$\delta = \max_{k \in \mathcal{K}} \frac{\gamma_k^2}{\langle f_l(\hat{\boldsymbol{P}}), \hat{\boldsymbol{v}}_k \rangle^2} - \max_{(\boldsymbol{V}, \boldsymbol{P}) \in \mathcal{V}(\boldsymbol{r}) \times \mathcal{P}} \min_{k \in \mathcal{K}} \frac{\gamma_k^2}{\langle \boldsymbol{v}_k, f_k(\boldsymbol{P}) \rangle^2}, \tag{12}$$

with $l = \arg\max_{k \in \mathcal{K}} \langle f_k(\hat{\boldsymbol{P}}), f_k(\hat{\boldsymbol{P}}) \rangle$.

An outline of the proof is given in the Appendix.

Proposition 3 provides bounds on the optimum value of the problem (4), under the assumption of bounded link performance functions. In the case of orthogonal networks, numerous performance metrics of common interest, such as the BER metric (1) and the MMSE metric (2), are bounded. The link performance functions (7), which have to be set in problem (4) to yield the max-min fairness problem for interference networks (Proposition 1) are bounded as well. Thus, (10) represents lower and upper bounds on link performance achieved under max-min fairness in orthogonal and interference networks. The Proposition allows for the following interpretation (by relative performance we mean now the ratio of performance requirement and performance value);

i.) The square of the worst relative link performance achieved under max-min fairness in a network is not larger than $\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)/c$. Hereby, $\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)$ is the Θ-function value of the network graph induced by a max-min fair network matrix $\hat{\boldsymbol{V}}$, and $\boldsymbol{w}, c$ are given by (11).

ii.) The square of the worst relative link performance achieved under max-min fairness in a network is at most by δ, given in (12), smaller than the value $\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w})/c$.

From (11) can be seen that the Θ-function of the network graph in the bounds (10) is weighted by the vector $\boldsymbol{w}^2$ of squared link performance requirements γ_k normalized by the corresponding coefficients r_k, $k \in \mathcal{K}$, which parameterize the set of allowable network matrices $\mathcal{V}(\boldsymbol{r})$. For an interference network, we can not give an easy interpretation of $\boldsymbol{w}^2$ (notice that in this case the performance requirements occurring in $\boldsymbol{w}$ are the inverse link sir requirements (8)). For an orthogonal network however, the parameter vector $\boldsymbol{r}/N$ groups the fractions of the resource set assigned to links over time. Thus, the interpretation of the weight vector $\boldsymbol{w}$ is as follows;

iii.) In an orthogonal network, the weight vector $\boldsymbol{w}$ in (11) contains the requirements on link performance recalculated for a single resource. In other words, for orthogonal networks the Θ-function in the bounds (10) is evaluated for the vector of squared per-resource requirements on link performance.

The scaling factor c in the bounds (10) corresponds to the largest 2-norm of a link performance function f_k, $k \in \mathcal{K}$, evaluated for a max-min fair power allocation. For interference networks, we need the specific setting (7) of the link performance function in the fairness problem (4), where ϵ is positive and arbitrarily small and $\hat{p}_{kk} \geq \epsilon > 0$, $k \in \mathcal{K}$. Thus, we yield for interference networks

$$c = \max_{k \in \mathcal{K}} \sum_{l \in \mathcal{K}, l \neq k} \frac{|h_{kl}|^2 \hat{p}_{ll}}{|h_{kk}|^2 \hat{p}_{kk}} + \sum_{k \in \mathcal{K}} \frac{\sigma_k^2}{\hat{p}_{kk}}.$$

A. Equality Characterization

The value δ is the maximum gap between the squared worst relative link performance under max-min fairness and the value of the Θ-function of the network graph induced by max-min fair network matrix. The first term in the expression of δ (12) is the maximum relative performance that would be achieved by a max-min fair network matrix, if the performance function of each link $k \in \mathcal{K}$ assumed the value $f_l(\hat{\boldsymbol{P}})$. Hence, we can formulate the following interpretation of the gap δ;

iv.) The value δ is the difference between the squared worst relative link performance under max-min fairness and the squared best relative link performance that would be achieved under a max-min fair network matrix and power allocation $\hat{\boldsymbol{V}}$, if the performance function of each link $k \in \mathcal{K}$ assumed the value with maximum 2-norm in $\mathcal{K}$ under a max-min fair power allocation $\hat{\boldsymbol{P}}$.

More abstractly, one can say that δ is a specific metric of imbalance in link performance occurring under max-min fairness. Precisely, if the dispersion of 2-norms of link performance function values under max-min fair power allocation is small, the bounds (10) are near each other. That is, the value $\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)/c$ is then a good approximation of the worst relative link performance under max-min fairness. From (12) can be seen that the value of the gap is determined merely by the sets $\mathcal{P}$ and $\mathcal{V}(\boldsymbol{r})$. This means that δ is specific for the constraints on power allocations and network matrices on hand. Fortunately, for some type of constraints δ becomes zero, which is our second main result.

Proposition 4: Let all assumptions of Proposition 3 be satisfied. Then, if $\mathcal{P} = \mathcal{P}_s$, we have $\delta = 0$ and thus

$$\max_{\substack{(\boldsymbol{V},\boldsymbol{P}) \in \\ \mathcal{V}(\boldsymbol{r}) \times \mathcal{P}_s}} \min_{k \in \mathcal{K}} \frac{\gamma_k^2}{\langle \boldsymbol{v}_k, f_k(\boldsymbol{P}) \rangle^2} = \frac{\Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)}{c}, \qquad (13)$$

with $\mathcal{V}(\boldsymbol{r}) \in \{\mathcal{A}(\boldsymbol{r}), \mathcal{B}(\boldsymbol{r})\}$, $\boldsymbol{r} \in \mathbb{R}_{++}^K$ and $\boldsymbol{w}, c$ defined in (11).

Proposition 4 says that the gap δ in (10) vanishes in the particular case of sum-constraint on the allocated link powers. This includes the case of orthogonal cellular downlink or cellular downlink with interference, for which we can formulate the following interpretation of Proposition 4;

v.) In a cellular downlink, the square of the worst relative link performance achieved under max-min fairness is equal to the scaled Θ-function value of the network graph induced by max-min fair network matrix, and with weight vector $\boldsymbol{w}$ given by (11). The scaling factor corresponds to the largest 2-norm of a link performance function f_k, $k \in \mathcal{K}$, evaluated for a max-min fair power allocation.

VI. General Discussion

The obtained network type-insensitive characterizations (10), (13) indicate that combinatorial (binary) graph properties of the interference coupling in an interference network and of the time sharing policy in an orthogonal network are decisive for link performance under max-min fairness. The characterizations imply also that graph description is a suitable universal description of seemingly different mechanisms, such as sharing of orthogonal resources and interference. Thus, one can say that, in broad terms, interference and time sharing of orthogonal resources are the same mechanisms from the point of view of max-min fairness.

The amount of information on the network contained in the Θ-function value of its network graph is quite small compared with the full network description by all its parameters. Nevertheless, Propositions 3, 4 show that this combinatorial information, and a single 2-norm value, are sufficient for the determination of (at least tight bounds on) worst relative link performance under max-min fairness, regardless of the network type. More precisely, (10) and (13) are determined merely by

i.) the spectral properties of some graph induced by max-min fair network matrix according to the orthogonal representation principle [14] (recall Proposition 2),

ii.) the 2-norm of a single value of a link performance function.

Feature i.) says, that the max-min fair network matrix (the max-min fair time sharing matrix or the interference matrix resulting, e.g., from the use of max-min fair sequences) influences the link performance only through the spectrum of its induced graph. Feature ii.) implies a similar conclusion, that values of link performance functions influence the (given tight bounds on) link performance under max-min fairness only through the maximum 2-norm under max-min fair power allocation.

To the best of our knowledge, general characterizations comparable with (10), (13) for the case of an orthogonal network with $N \geq K$ and for the case of an interference network with $\sigma_k^2 > 0$, $k \in \mathcal{K}$, do not exist. However, for the special case of an interference-dominated network ($\sigma_k^2 = 0$, $k \in \mathcal{K}$), we can relate our results to the known results on power control for CDMA networks [8], [3]. For instance, a known result is that, given $\sigma_k^2 = 0$, $k \in \mathcal{K}$, we have [3]

$$\sup_{\boldsymbol{P} \in \mathcal{P} \cap \mathbb{R}_{++}^{K \times K}} \min_{k \in \mathcal{K}} \frac{\mathrm{SIR}_k(\boldsymbol{B}, \boldsymbol{P})}{\hat{\gamma}_k} = \frac{1}{\lambda_{max}(Diag(\hat{\gamma})\boldsymbol{D}(\boldsymbol{B}))}, \quad (14)$$

with the matrix function $\boldsymbol{D}(\boldsymbol{B}) = [b_{kl}|h_{kl}|^2/|h_{kk}|^2]$, $\boldsymbol{B} = (b_{kl}) \in \mathcal{B}(\boldsymbol{r})$, $\boldsymbol{r} \in \mathbb{R}_{++}^K$. Clearly, (14) is satisfied in particular for the max-min fair interference matrix $\hat{\boldsymbol{B}}$, which solves (6). Thus, with Propositions 1, 4 we yield for $\mathcal{P} = \mathcal{P}_s$ that

$$\frac{1}{\lambda_{max}(Diag(\hat{\gamma})\boldsymbol{D}(\hat{\boldsymbol{B}}))^2} = \frac{\Theta(G(\hat{\boldsymbol{B}}), \boldsymbol{w}^2)}{c}, \quad (15)$$

where $\boldsymbol{w}, c$ are given by (11) and vector $\boldsymbol{\gamma}$ is obtained from $\hat{\boldsymbol{\gamma}}$ by (8). In this sense, Propositions 1, 4 can be seen as the extension/ generalization of the results on max-min fairness in interference-dominated networks to the case of general interference networks and orthogonal networks.

The results of this work indicate that graph-related algorithmic tools might possibly be an aid in the design of novel algorithms finding the max-min fair power allocation and max-min fair network matrix. For instance, from the structure of the unifying problem (4) we can conclude that an alternating algorithm design, of the type proposed e.g. in [7], is likely to converge. Applied to (4), such algorithm type consists in the alternation of iteration steps optimizing the power allocations and network matrices, respectively. The obtained characterizations suggest to observe after each iteration $i \in \mathbb{N}$ the values $c(i) = \max_{k \in \mathcal{K}} \langle f_k(\boldsymbol{P}(i)), f_k(\boldsymbol{P}(i)) \rangle$ and $\Theta(G(\boldsymbol{V}(i)), \boldsymbol{w}^2)$ (or at least some approximation of the latter one [15]). Then, according to (10), (13), the ratio $\Theta(G(\boldsymbol{V}(i+1)), \boldsymbol{w}^2)c(i)/(c(i+1)\Theta(G(\boldsymbol{V}(i)), \boldsymbol{w}^2))$ might serve as an indication of the distance to the optimum at $i+1$-th iteration and can give rise to some exit condition.

VII. Appendix - (Sketch of) Proof of Proposition 3

Define $\phi(\boldsymbol{V}, \boldsymbol{P}) = \min_{k \in \mathcal{K}} \gamma_k / \langle \boldsymbol{v}_k, f_k(\boldsymbol{P}) \rangle$. and the vectors $\boldsymbol{z}_k = w_k(f_l(\hat{\boldsymbol{P}})/\sqrt{c} - \sqrt{c}\hat{\boldsymbol{v}}_k / \langle f_l(\hat{\boldsymbol{P}}), \hat{\boldsymbol{v}}_k \rangle)$, $k \in \mathcal{K}$. We have then $\langle \boldsymbol{z}_j, \boldsymbol{z}_k \rangle = -w_j w_k$ iff $j \neq k$ and pair (j,k) is nonadjacent in $G(\hat{\boldsymbol{V}})$, and $\langle \boldsymbol{z}_k, \boldsymbol{z}_k \rangle \leq -w_k^2 + c\gamma_k^2 / \langle f_l(\hat{\boldsymbol{P}}), \hat{\boldsymbol{v}}_k \rangle^2$. Then, defining $\boldsymbol{Z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K)$ we yield $\boldsymbol{Z}'\boldsymbol{Z} \preceq -\boldsymbol{C} + Diag(c\gamma_1^2/\langle f_l(\hat{\boldsymbol{P}}), \hat{\boldsymbol{v}}_1 \rangle^2, \ldots, c\gamma_K^2/\langle f_l(\hat{\boldsymbol{P}}), \hat{\boldsymbol{v}_K} \rangle^2)$, where $\boldsymbol{C} \in \mathcal{C}(G(\hat{\boldsymbol{V}}), \boldsymbol{w})$ (see Proposition 2). The addition of $c(\phi^2(\hat{\boldsymbol{V}}, \hat{\boldsymbol{P}}) + \delta)\boldsymbol{I}$ and some manipulations yield then $c(\phi^2(\hat{\boldsymbol{V}}, \hat{\boldsymbol{P}}) + \delta)\boldsymbol{I} - \boldsymbol{C} \succeq 0$ if $\delta = \max_{k \in \mathcal{K}} c\gamma_k^2 / \langle f_l(\hat{\boldsymbol{P}}), \hat{\boldsymbol{v}}_k \rangle^2 - \phi^2(\hat{\boldsymbol{V}}, \hat{\boldsymbol{P}})$. But this implies $\phi^2(\hat{\boldsymbol{V}}, \hat{\boldsymbol{P}}) \geq \lambda_{max}(\boldsymbol{C})/c - \delta$, and with $\lambda_{max}(\boldsymbol{C}) \geq \min_{\boldsymbol{C} \in \mathcal{C}(G(\hat{\boldsymbol{V}}), \boldsymbol{w})} \lambda_{max}(\boldsymbol{C}) = \Theta(G(\hat{\boldsymbol{V}}), \boldsymbol{w}^2)$, by Proposition 2, proves the lower bound.

To prove the upper bound, take $\hat{\boldsymbol{C}} = \arg\min_{\boldsymbol{C} \in \mathcal{C}(G(\hat{\boldsymbol{V}}), \boldsymbol{w})} \lambda_{max}(\boldsymbol{C})$. Then, one can show that there exists $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K) \in \mathbb{R}^{N \times K}$, such that $c(\lambda_{max}(\hat{\boldsymbol{C}})) - \hat{\boldsymbol{C}}) = \boldsymbol{X}'\boldsymbol{X}$, and such that $\boldsymbol{x}_k$, $k \in \mathcal{K}$ have a zero entry on the last position. Using the fact that for $\boldsymbol{h} = (0, \ldots, 0, \sqrt{c})$ we have $\langle \boldsymbol{h}, \boldsymbol{x}_k \rangle = 0$, $k \in \mathcal{K}$ we define a specific matrix $\boldsymbol{U} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_K)'$, with $\boldsymbol{u}_k = \boldsymbol{u}_k(\boldsymbol{h}, \boldsymbol{x}_k)$ such that $\langle \boldsymbol{h}, \boldsymbol{u}_j \rangle / \gamma_j = \sqrt{c/\lambda_{max}(\hat{\boldsymbol{C}})}$, and with $l_j = \|\boldsymbol{u}_j\|_1$ also $l_j \leq r_j$, $j \in \mathcal{K}$. Using the parameterized set $\mathcal{D}_i(\boldsymbol{x}) = \{\boldsymbol{D} \in \mathbb{R}_+^{K \times N} : \|\boldsymbol{d}_k\|_i = x_k, k \in \mathcal{K}\}$, $i = 1, 2$, $\boldsymbol{x} \in \mathbb{R}_{++}^K$, these features of the vectors $\boldsymbol{u}_j$, $j \in \mathcal{K}$ yield

$$\lambda_{max}(\hat{\boldsymbol{C}})/c \geq \min_{(\boldsymbol{D}, \boldsymbol{E}) \in \delta\mathcal{D}_1(\boldsymbol{l}) \times \delta\mathcal{D}_2(c\boldsymbol{1})} \max_{k \in \mathcal{K}} \frac{\gamma_k^2}{\langle \boldsymbol{e}_k, \boldsymbol{d}_k \rangle^2}, \quad (16)$$

due to $\boldsymbol{U} \in \delta\mathcal{D}_1(\boldsymbol{l}) \subseteq \mathcal{D}_2(\boldsymbol{r})$ and $(\boldsymbol{h}, \ldots, \boldsymbol{h})' \in \delta\mathcal{D}_2(c\boldsymbol{1})$. Further, with the minmax-maxmin inequality, some convex-theoretic arguments and the fact that $(f_1(\hat{\boldsymbol{P}}), \ldots, f_K(\hat{\boldsymbol{P}}))' \in \delta\mathcal{D}_2(c\boldsymbol{1})$, we obtain $\min_{(\boldsymbol{D}, \boldsymbol{E}) \in \delta\mathcal{D}_1(\boldsymbol{l}) \times \delta\mathcal{D}_2(c\boldsymbol{1})} \max_{k \in \mathcal{K}} \gamma_k^2 / \langle \boldsymbol{e}_k, \boldsymbol{d}_k \rangle^2 \geq \max_{(\boldsymbol{D}, \boldsymbol{E}) \in \delta\mathcal{D}_1(\boldsymbol{l}) \times \delta\mathcal{D}_2(c\boldsymbol{1})} \min_{k \in \mathcal{K}} \gamma_k^2 / \langle \boldsymbol{e}_k, \boldsymbol{d}_k \rangle^2 \geq \phi^2(\hat{\boldsymbol{V}}, \hat{\boldsymbol{P}})$. By the definition of $\hat{\boldsymbol{C}}$, Proposition 2 and (16) this completes the proof.

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